SPR and REP Improved Techniques in Hexahedrons

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Abstract: In this work it is presented the application of the smoothed stress field improved techniques for superconvergent patch recovered (SPR) and recovery by equilibrium in patches (REP) on hexahedrons in problems with three-dimensional domain that allows to obtain more precise results when using the method of the finite elements. In order to analyze the behavior of each technique separately, a h_adaptative procedure is given, where the estimator of Zienkiewicz-Zhu error (z^2) is used for the estimation of the error that involves the stress range improved. In this, the effectiveness index for problems with smooth solution and cases where the solution is singular are analyzed. The obtained results show that both techniques are computationally analogous when the meshes have elevated degrees of freedom (dof), also it is observed a similar behavior when the effectiveness index is analyzed.

Key-Words: superconvergent stress recovery, superconvergent patch recovery technique, recovery by equilibrium in patches, equilibrium constraint.

1 Introduction

The Superconvergent Patch Recovered (SPR) technique is one of the simplest and popular techniques with low computacional cost. In fact, many other techniques are *dedicated* to include improvements to the original SPR technique like the ones made by: Lee and Lo [1], Blacker et al [2], Wiberg et al [3], and Ródenas [4]. The improvement proposed by Lee and Lo is based in assuring the robustness and stability of the SPR technique, manipulating the matrices of present stress smoothed in the method; whereas the improvements proposed by the other authors are based on using the available information in the contour for increasing the precision of the nodal values of the stress range smoothed in this zone of the analyzed component.

Other technique as the one proposed by Boroomand and Zienkiewicz [5, 6], suggests a new procedure known as "Recovery by Equilibrium in patches (REP)" for the construction of smoothing stress derived from the solution of finite elements. This one differs from the SPR in which it forces to the fulfillment of the balance in patch and it does not require of the previous knowledge of superconvergence points, the computacional cost is according to its authors, superior to SPR but inferior to the previously mentioned improvements proposed by Blacker and Wiberg. From this, it is observed how techniques as SPR and REP have been of great impact in the smoothed of stress and is for that reason that in this work appears an implementation of these improved techniques to hexahedrons.

This article has been organized in five sections. In section 2, improved SPR and REP techniques are presented in order to obtain the smoothing stress range. In section 3 the problems in numerical solution are described and presented, for validation of results. In section 4, the analyses of results are given. Finally, section 5 contains the most important conclusions in the implementation of these techniques in three-dimensional dominions.

2 Improved SPR and REP Techniques

2.1 Improved Superconvergent Patch Recovered (SPR)

The objective of SPR is to find the vector of smoothed nodal stress, s^* so that the stress inside each element are defined through the same interpolation functions used for the displacements interpolation.

In SPR technique, the polynomic adjustment used in each patch, is written as:

$$s^*_p = pa \tag{1}$$

where p contains the appropriate terms to complete a same order polynomial that the form functions, and

a is a vector of unknown parameters corresponding to the polynomial coefficients.

The determination of the coefficients a_i is given by means of adjustment of Minimum Square of the finite elements values. For it, a function of next form can be raised:

$$F(a) = \sum_{i=1}^{Nptg} \left(s_{ef} - s_p^* \right)^2_{(x_i,y_i,z_i)} = \sum_{i=1}^{Nptg} \left(s_{ef} - \boldsymbol{p}\,a \right)^2_{(x_i,y_i,z_i)} \qquad (2)$$

where $s_{ef\ (x_i,y_i,z_i)}$ are the evaluated stress of finite elements in the coordinates (x_i, y_i, z_i) of the numerical integration points, and Nptg is the total points number of numerical integration in patch. Minimizing $F(a)$ it is obtained:

$$\sum_{i=1}^{Nptg} \left(p^T p \right)_{(x_i,y_i,z_i)} a = \sum_{i=1}^{Nptg} \left(p^T s_{ef} \right)_{(x_i,y_i,z_i)} \qquad (3)$$

that can be solved in matrix form as:

$$a = \mathbf{A}^{-1} b \qquad (4)$$

where $\mathbf{A}$ and $\boldsymbol{b}$ are:

$$\mathbf{A} = \sum_{i=1}^{Nptg} \left(p^T p \right)_{(x_i,y_i,z_i)}, \quad \boldsymbol{b} = \sum_{i=1}^{Nptg} \left(\boldsymbol{p}^T s_{ef} \right)_{(x_i,y_i,z_i)} \qquad (5)$$

Once the vector a is found, the calculation of the fit stress, s_p^* in a node, is immediate found replacing in equation (1) the coordinates of each inner node of the patch.

The SPR-R technique has been proponed by Ródenas [4] for bidimensional problems and it assures the exact fulfillment of the tension in contour nodes imposed restrictions. This one can be extended to three-dimensional problems [7] using an analogous procedure to the developed by this author. In this, the interpolation polynomial for stress in patch (s_p^*) and each of its components ($s_x, s_y, s_z, s_{xy}, s_{xz}, s_{yz}$) can be expressed as:

$$s_{pi}^* = a_{i1} + a_{i2}x + a_{i3}y + a_{i4}z + a_{i5}x^2 + a_{i6}y^2 + \\ + a_{i7}z^2 + \ldots \quad i = x,\ y,\ z,\ xy,\ xz,\ yz \qquad (6)$$

The exact fulfillment of the imposed restrictions of tension in the nodes of patch assembling can be forced incorporating in the equations system to be solved, the known values for these restrictions. For this, in the patches of nodes located in the contour of the stress interpolation polynomials (s_p^*) will be raised in a system of local axes reference x, h and z with the origin in the assembling node for the patch and with the axes oriented according to the directions normal and tangential to the contour.

In order to raise the equations of SPR technique in the new system of coordinates it is made a system of coordinate's transfer and then two turns, one in y and then in x. Therefore, the local coordinates (x, h, z) corresponding to a point of the global coordinates (x, y, z) will be evaluated using the expression:

$$\begin{Bmatrix} ? \\ ? \\ ? \end{Bmatrix} = \mathbf{Rot} \begin{Bmatrix} x - x_n \\ y - y_n \\ z - z_n \end{Bmatrix} \qquad (7)$$

where (x_n, y_n, z_n) are the global coordinates of the assembling node of the patch and the rotation matrix is given by:

$$\mathbf{Rot} = \begin{bmatrix} \cos f & 0 & -sen f \\ -sen?en?f & \cos ? & -sen?\cos f \\ \cos ?sen f & sen? & \cos ?\cos f \end{bmatrix} \qquad (8)$$

angles f, q define the normal direction of the contour with respect to the global axes x, y, z.

The new reference system is comfortable to force the exact fulfillment of the imposed restrictions of tension in the assembling nodes of the patch, since the normal stress to the contour are known.

2.2 Improved Recovery by Equilibrium Patches (REP)

This procedure consists of balancing the stress improved in patch in the same form that in the MEF (therefore the obtained stress satisfy the discreet balance conditions) and with an analogous procedure to the used in the SPR, where the stress in each patch is approximated with a polynomial of appropriate order s_p^*.

Assuming the continuous form of the tension on patch, it can be expressed analogous like in the equation (1) if $s = s_p^* = \mathbf{P}a$:

$$\left(\int_{\Omega_p} \mathbf{B}^T P\, d\Omega \right) a = \left(\int_{\Omega_p} \mathbf{B}^T \mathbf{DB}\, d\Omega \right) u^* \qquad (9)$$

Making $\mathbf{H} = \left(\int_{\Omega_p} \mathbf{B}^T \mathbf{P}\, d\Omega \right)$ and $\left(\int_{\Omega_p} \mathbf{B}^T DB\, d\Omega \right) u^* = Fp = \int_{\Omega_p} \mathbf{B}^T s_{ef}\, d\Omega$; Equation (9) can written as:

$$\mathbf{H}a = F_p \qquad (10)$$

This equation can be only satisfied approximately; therefore it is necessary to apply minimum square, for which the following functional, analog to the raised in SPR technique, is used:

$$F(a) = (\mathbf{H}a - F_p)^T (\mathbf{H}a - F_p) \qquad (11)$$

Then, for finding the coefficients a , it is enough with imposing $\frac{\partial F(a)}{\partial a} = 0$, for finding:

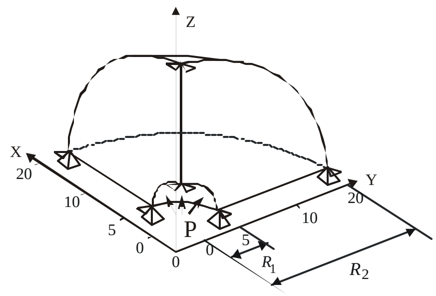
$$a = \left[\mathbf{H}^T \mathbf{H}\right]^{-1} \mathbf{H}^T F_p \qquad (12)$$

In order to improve the solution in the border, the extension of the SPR-R improvement, presented by Ródenas, is used; that assures the exact fulfillment.

3 Cases for Numerical Study

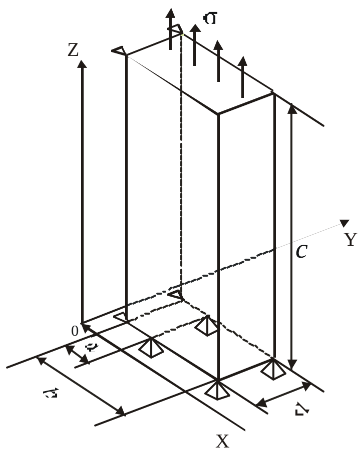
For the numerical analyses, it will be used a model that corresponds to $1/8$ of an sphere, imposing the contour conditions as in the Fig. 1 and loaded Plate tension with lateral crack as it is in Fig. 2.

In order to make the study of the previously mentioned techniques behavior at local level it is used for the sphere hexahedric linear elements (the quadratic elements will be studied later) with an initial mesh of 20 elements, in which the contour nodes are distributed as it is shows in Fig. 3a.



$R_1 = 5$
$R_2 = 20$
$P = 1$
$E = 1000$
$u = 0.3$
$\|\mathbf{u}_{ex}\|^2 = 0.130899$

Fig. 1.- Sphere with inner pressure



$s = 1000$
$a = 0.6$
$b = 2$
$c = 6$
$d = 1$
$E = 10^7$
$u = 0.333$
$\|\mathbf{u}_{ex}\|^2 = 0.112007$

Fig. 2.- Plate with lateral crack

For the stress ranges shown in Fig. 3b, it is observed that the behavior of both techniques is almost identical since they present quite near values of tension and they almost agree with the theoretical values of tension.

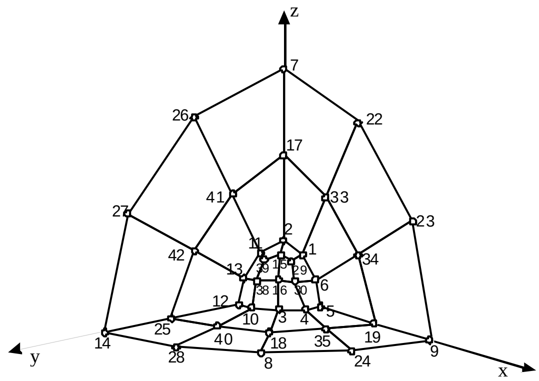


Fig. 3a.- Sphere initial mesh.

It is important now to establish the global behavior of these techniques through the effectiveness index; therefore it will be presented in the following section the obtained results for linear and quadratic elements in the sphere and the crack, using first the SPR-R technique and then the improved-REP procedure.

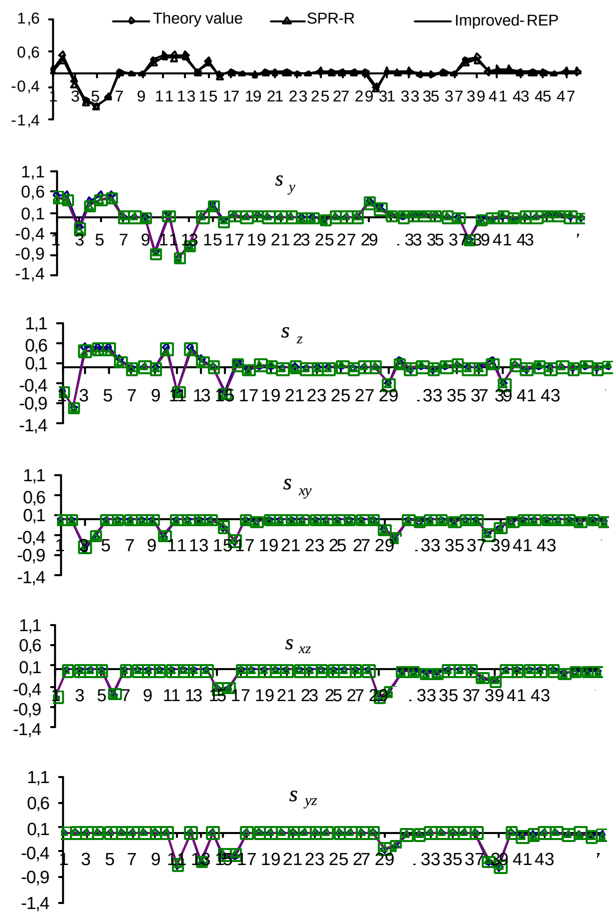
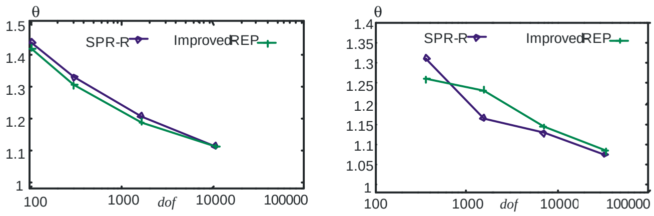


Fig. 3b.- s^* in the sphere dominion nodes.

4 Results

The implemented procedures for the reconstruction of the stress range present a similar behavior, since for each case; the magnitudes of the effectiveness index tend to the unit when they increase the degrees of freedom.

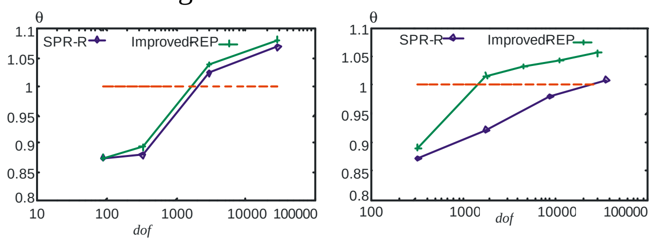
In the problem of the sphere exposed to internal pressure when linear elements are used, the value of q closest to the unit is found using the improved-REP. In fact, in any of the mesh steps, the magnitudes of the effectiveness indices are always near of the unitary value, as it is shown in Figure 4. When quadratic elements are used, the behavior of the second mesh is inverse. In fact, from the second mesh step, the values of effectiveness that presents the SPR-R technique are nearer to the unitary value, as it is shown in Fig. 4.



(a) Linear Elements (b) Quadratic Elements

Fig. 4.- Effectiveness of SPR-R and improved-REP in the sphere

For the problem of the plate with lateral crack with linear elements, the improved-REP and SPR-R techniques present an analogous behavior, where the effectiveness indices for the SPR-R technique are nearer to the unitary value. A similar behavior exhibits the quadratic elements, where it is possible to be observed that the SPR-R technique presents better effectiveness indices than improved-REP, as can be observed in Fig. 5.



(a) Linear Elements (b) Quadratic Elements

Fig. 5.- Effectiveness of SPR-R and improved-REP in the crack

5 Conclusion

The local level behavior of the SPR-R and Improved-REP techniques show that adding the contour conditions, it will be remarkably improved the nodal stress value in this zone.

In general, from the analyzed examples for each of the developed techniques for the stress smoothed, it is possible to say that:

For the sphere case, improved-REP presents the effectiveness magnitudes nearer to the unit if linear elements are used (see Fig. 4). With quadratic elements the SPR-R technique exhibits effectiveness indices next to the unitary value. When the problem with singular solution is considered, it is preferable to use the SPR-R procedure, since with this; the unitary effectiveness index is reached with less degrees or freedom, staying in successive refinements as it is shown in Fig. 5. Therefore, in general, analogous results are obtained when applying the improved SPR-R and REP technique.

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